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2009 J. Phys. A: Math. Theor. 42 025201

(http://iopscience.iop.org/1751-8121/42/2/025201)

J. Phys. A: Math. Theor. **42** (2009) 025201 (11pp) doi:10.1088/1751-8113/42/2/025201

A combinatorial method for the vanishing of the Poisson brackets of an integrable Lotka–Volterra system

Yoshiaki Itoh[1]

The Institute of Statistical Mathematics and the Graduate University for Advanced Studies, 4-6-7 Minami-Azabu Minatoku, Tokyo 106-8569, Japan

E-mail: itoh@ism.ac.jp

Received 6 April 2008, in final form 23 October 2008
Published 25 November 2008
Online at stacks.iop.org/JPhysA/42/025201

Abstract

The combinatorial method is useful to obtain conserved quantities for some nonlinear integrable systems, as an alternative to the Lax representation method. Here we extend the combinatorial method and introduce an elementary geometry to show the vanishing of the Poisson brackets of the Hamiltonian structure for a Lotka–Volterra system of competing species. We associate a set of points on a circle with a set of species of the Lotka–Volterra system, where the dominance relations between points are given by the dominance relations between the species. We associate each term of the conserved quantities with a subset of points on the circle, which simplifies to show the vanishing of the Poisson brackets.

PACS numbers: 02.10.Ox, 02.30.Ik, 87.23.−n

1. Periodic Lotka–Volterra system

The Toda lattice is known to have soliton solutions [1]. The periodic Toda lattice is a typical nonlinear integrable system, which has m conserved quantities [2–4] for $2m$ variables. The quantities are obtained by using the Lax representation [3, 4] as well as by using a combinatorial method [2]. Another typical system is a periodic Lotka–Volterra system of competing species

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = P_i \left(\sum_{j=1}^{s} P_{i-j} - \sum_{j=1}^{s} P_{i+j} \right) \tag{1}$$

for relative abundances $P_i, i = 1, 2, \ldots, m$. The system (1) with $s = 1$ is known to be an integrable discretization of Korteweg de Vries equation [5–8]. The Lax representation is

[1] Address for correspondence: 1588-41 Koshigoe, Kamakura 248-0033, Japan.

obtained for the general integer $s, 0 < s < \frac{m}{2}$, by Bogoyavlensky [7–11], which gives the conserved quantities as in the case of the Toda lattice. For the non-periodic case of infinite m, soliton solutions are known as in the case of the Toda lattice [12, 13].

The interpretation of Hamiltonian structures and Lax representations in the framework of the r-matrix theory [14–18] is useful to study the Toda lattice, the Lotka–Volterra system equation (1) and other related problems. Extending the study [17] of $s = 1$, the vanishing of the Poisson brackets for the Hamiltonian structure of equation (1) was naturally shown in [19] considering the r-matrix for the Lax representation [8, 18], which is the solution of the Yang–Baxter equation.

Hamiltonian structure of Lotka–Volterra systems gives various interesting problems [20–22]. Here we restrict our attention to equation (1) and consider the Poisson structure [8] as in section 2. For the case $m = 2s + 1$ for equation (1), the $s + 1$ conserved quantities are given by using a combinatorial method [25]. The quantities are obtained from a deterministic approximation of the quantities (martingales) for a stochastic model of competing species [23, 24] . We extend the combinatorial method [25] in section 3 to introduce an elementary geometry [26–28], which gives another approach to the vanishing of the Poisson brackets for the case $m = 2s + 1$. We associate a set of points on a circle with a set of species of the Lotka–Volterra system, where the dominance relations between points are given by the dominance relations between the species. We associate each term of the conserved quantities with a subset of points on the circle, which simplifies the argument to show the vanishing of the Poisson brackets as given in section 4. In section 5 we discuss possible problems to apply our method. We briefly show that our combinatorial method is obtained to calculate the asymptotic probability of coexistence of species in a population [24], as in a problem of population genetics [32].

2. Conserved quantities and Hamiltonian structure

The conserved quantities of equation (1), for positive integers m and $s < \frac{m}{2}$, are obtained systematically from the Lax representation by Bogoyavlensky [8]. For example the first three conserved quantities [19] are

$$I_1 = \sum_{i=1}^{m} P_i, \tag{2}$$

$$I_2 = \sum_{i=1}^{m} \left(\frac{1}{2} P_i^2 + P_i \sum_{j=1}^{s} P_{i+j} \right), \tag{3}$$

$$I_3 = \sum_{i=1}^{m} \left(\frac{1}{3} P_i^3 + P_i \sum_{j=1}^{s} P_{i+j} \sum_{l=0}^{j+s} P_{i+l} \right). \tag{4}$$

Here we introduce another system of conserved quantities by using a combinatorial method. The dynamical system, equation (1), has the Poisson structure [8], and we apply the structure to our case $m = 2s + 1$. We show the vanishing of the Poisson bracket for our system of conserved quantities [25].

Define a_{ij} by the equation

$$\sum_{j=1}^{2s+1} a_{ij} P_j \equiv \sum_{k=1}^{s} P_{i-k} - \sum_{k=1}^{s} P_{i+k}. \tag{5}$$

We say the species i dominates the species j ($j \prec i$) if $a_{ij} = 1$. If $a_{ij} = -1$, we say the species i is dominated by the species j ($i \prec j$). Consider $2\alpha + 1$ species out of the $2s + 1$ species. If each of the $2\alpha + 1$ species dominates the other α species and is dominated by the other remaining α species, then we say the $2\alpha + 1$ species are in a regular tournament of order $2\alpha + 1$. Take $2\alpha + 1$ individuals (particles) at random from the system. Let G_α be the probability that the corresponding $2\alpha + 1$ species of the $2\alpha + 1$ particles are in a regular tournament, then the G_α, $\alpha = 0, 1, 2, \ldots, s$, are conserved quantities [25], that is to say,

$$\frac{\mathrm{d}}{\mathrm{d}t} G_\alpha = 0. \tag{6}$$

For example, for the case $2s + 1 = 5$, we have the conserved quantities,

$$G_0 = P_1 + P_2 + P_3 + P_4 + P_5, \tag{7}$$

$$G_1 = P_1P_2P_4 + P_2P_3P_5 + P_3P_4P_1 + P_4P_5P_2 + P_5P_1P_3, \tag{8}$$

$$G_2 = P_1P_2P_3P_4P_5. \tag{9}$$

Put $v_i = \log(P_i)$ for $i = 1, 2, \ldots, 2s + 1$ and Λ as the skew-symmetric operator with the entries

$$\lambda_{i,i-k} = 1, \qquad \lambda_{i,i+k} = -1, \qquad k = 1, 2, \ldots, s.$$

Equation (1) takes the form of the Poisson structure,

$$\dot{v}_i = \sum_{k=1}^{s} \exp[v_{i-k}] - \sum_{k=1}^{s} \exp[v_{i+k}] = \sum_{k=-s}^{s} \lambda_{ik} \exp[v_{i+k}] = \sum_{j=1}^{2s+1} \lambda_{ij} \frac{\partial H}{\partial v_j}, \tag{10}$$

with the Hamiltonian

$$H = \sum_{j=1}^{2s+1} \exp[v_j]. \tag{11}$$

Putting $P_i = \exp[v_i]$ for $i = 1, 2, \ldots, 2s + 1$ into G_α, we get the conserved quantities F_α, $\alpha = 0, 1, \ldots, s$, of equation (10). For example for $s = 2$,

$$F_0 = \exp[v_1] + \exp[v_2] + \exp[v_3] + \exp[v_4] + \exp[v_5], \tag{12}$$

$$\begin{aligned} F_1 = \exp[v_1 + v_2 + v_4] + \exp[v_2 + v_3 + v_5] + \exp[v_3 + v_4 + v_1] \\ + \exp[v_4 + v_5 + v_2] + \exp[v_5 + v_1 + v_3], \end{aligned} \tag{13}$$

$$F_2 = \exp[v_1 + v_2 + v_3 + v_4 + v_5]. \tag{14}$$

Note that each of the conserved quantities is not the Casimir function. We give a proof of the following theorem of the vanishing of the Poisson brackets for equation (10).

Theorem

$$\{F_q, F_r\} = \sum_{i,j=1}^{2s+1} \lambda_{ij} \frac{\partial F_q}{\partial v_i} \frac{\partial F_r}{\partial v_j} = 0, \tag{15}$$

for $q, r = 1, 2, \ldots, s$.

3. Configurations of points on a circle

Consider two points on a unit circle, X and Y, whose coordinates are $(\cos x, \sin x)$ and $(\cos y, \sin y)$, $0 \leqslant x, y < 2\pi$, respectively. If the counterclockwise way from X to Y on the circle is shorter than the clockwise way, we denote $X \prec Y$. If the counterclockwise way from Y to X on the circle is shorter than the clockwise way, we denote $Y \prec X$. We define $I_{X,Y} = 1$ for $Y \prec X$, $I_{X,Y} = -1$ for $X \prec Y$, and $I_{X,Y} = 0$ otherwise. Let us denote the shorter arc with end points X, Y, by the arc $[X, Y]$.

Consider a unit circle. Take $2s+1$ diameters, $j = 1, \ldots, 2s+1$, arbitrarily on it. Number the $2(2s+1)$ ends (points) of the diameters in counterclockwise way starting from an arbitrary end (point). Name the points $1, 3, 5, \ldots, 2(2s+1)-1$ as $W_1, W_2, \ldots, W_{2s+1}$ respectively, and $2, 4, 6, \ldots, 2(2s+1)$ as $w_1, w_2, \ldots, w_{2s+1}$ respectively. Let us denote the set of points as $W = \{W_1, W_2, \ldots, W_{2s+1}\}$ and $w = \{w_1, w_2, \ldots, w_{2s+1}\}$. The $2s+1$ points (species) $W_1, W_2, \ldots, W_{2s+1}$ are in a regular tournament of order $2s+1$, as $W_j \prec W_{j+1}, W_{j+2}, \ldots, W_{j+s}$ and $W_{j-1}, W_{j-2}, \ldots, W_{j-s} \prec W_j$, for $j = 1, 2, \ldots, 2s+1$. Each of the $2s+1$ points (species) W_i have variable $P_i(t)$ for $i = 1, 2, \ldots, 2s+1$, which evolves by equation (1) with $m = 2s+1$. We consider the two subsets of the $2s+1$ diameters as given in the following I, II and III.

Take the ends of $2q+1$ diameters out of the above $2s+1$ diameters in counterclockwise way as $A_1, A_2, \ldots, A_{2q+1}$, where $\{A_1, A_2, \ldots, A_{2q+1}\} = A \subset W$. Take the ends of $2r+1$ diameters out of the above $2s+1$ diameters in counterclockwise way as $B_1, B_2, \ldots, B_{2r+1}$, where $\{B_1, B_2, \ldots, B_{2r+1}\} = B \subset W$. Then the following I, II and III are mutually equivalent.

(I) $a_1 \prec A_1 \prec a_2 \prec A_2 \cdots \prec a_j \prec A_j \prec a_{j+1} \cdots a_{2q+1} \prec A_{2q+1} \prec a_1$, where $\{a_1, a_2, \ldots, a_{2q+1}\} = a \subset w$ and a_{j+q+1} is the other end of the diameter with the end A_j for $j = 1, 2, \ldots, 2q+1$.

$b_1 \prec B_1 \prec b_2, \prec B_2 \ldots \prec b_j \prec B_j \prec b_{j+1} \ldots b_{2r+1} \prec B_{2r+1} \prec b_1$, where $\{b_1, b_2, \ldots, b_{2r+1}\} = b \subset w$ and b_{k+r+1} is the other end of the diameter with the end B_k for $k = 1, 2, \ldots, 2r+1$.

(II) $A_j \prec A_{j+1}, A_{j+2}, \ldots, A_{j+q}$ and $A_{j-1}, A_{j-2}, \ldots, A_{j-q} \prec A_j$, for $j = 1, 2, \ldots, 2q+1$.

$B_k \prec B_{k+1}, B_{k+2}, \ldots, B_{k+r}$ and $B_{k-1}, B_{k-2}, \ldots, B_{k-r} \prec B_k$, for $k = 1, 2, \ldots, 2r+1$.

(III) The points $A_1, A_2, \ldots, A_{2q+1}$, are in a regular tournament of order $2q+1$.

The points $B_1, B_2, \ldots, B_{2r+1}$ are in a regular tournament of order $2r+1$.

Assume there exist exactly $0 < \gamma$ arcs $[A_{j_l}, B_{k_l}]$, $A_{j_l} \in A$ and $B_{k_l} \in B$, $l = 1, 2, \ldots, \gamma$ with no point of $a \cup A \cup b \cup B$ in the arc $[A_{j_l}, B_{k_l}]$ other than A_{j_l}, B_{k_l}.

We call $B'_{k_l} = A_{j_l}$ and $A'_{j_l} = B_{k_l}$ for each pair of points $\{A_{j_l}, B_{k_l}\}$, $l = 1, 2, \ldots, \gamma$, $A'_j = A_j$ for A_j which does not coincide with any one of A_{j_l}, $l = 1, 2, \ldots, \gamma$, for $j = 1, 2, \ldots, 2q+1$, and $B'_k = B_k$ for B_k which does not coincide with any one of B_{k_l}, $l = 1, 2, \ldots, \gamma$, for $k = 1, 2, \ldots, 2r+1$.

Let us call the pair of sets $A' = \{A'_1, A'_2, \ldots, A'_{2q+1}\}$ and $B' = \{B'_1, B'_2, \ldots, B'_{2r+1}\}$, the conjugate pair of sets of the above pair of sets, $A = \{A_1, A_2, \ldots, A_{2q+1}\}$ and $B = \{B_1, B_2, \ldots, B_{2r+1}\}$.

We see,

(I′) $a_1 \prec A'_1 \prec a_2 \prec A'_2 \cdots \prec a_j \prec A'_j \prec a_{j+1} \cdots a_{2q+1} \prec A'_{2q+1} \prec a_1$, where a_j is the other end of the diameter with the end A_{j+q} for $j = 1, 2, \ldots, 2q+1$ and $\{a_1, a_2, \ldots, a_{2q+1}\} = a \subset w$.

$b_1 \prec B'_1 \prec b_2, \prec B'_2 \ldots \prec b_j \prec B'_j \prec b_{j+1} \ldots b_{2r+1} \prec B'_{2r+1} \prec b_1$, where b_k is the other end of the diameter with the end B_{k+r} for $k = 1, 2, \ldots, 2r+1$ and $\{b_1, b_2, \ldots, b_{2r+1}\} = b \subset w$.

(II′) $A'_j \prec A'_{j+1}, A'_{j+2}, \ldots, A'_{j+q}$ and $A'_{j-1}, A'_{j-2}, \ldots, A'_{j-q} \prec A'_j$, for $j = 1, 2, \ldots, 2q+1$. $B'_k \prec B'_{k+1}, B'_{k+2}, \ldots, B'_{k+r}$ and $B'_{k-1}, B'_{k-2}, \ldots, B'_{k-r} \prec B'_k$, for $k = 1, 2, \ldots, 2r+1$.

(III′) The points of the set $A' = \{A'_1, A'_2, \ldots, A'_{2q+1}\}$ are in a regular tournament of order $2q+1$.
The points of the set $B' = \{B'_1, B'_2, \ldots, B'_{2r+1}\}$ are in a regular tournament of order $2r+1$.

We see that $(A')' = A$, $(B')' = B$ and $A \cup B = A' \cup B'$. The conjugate pair of sets A and B is a pair of sets A' and B', and the conjugate pair of sets A' and B' is a pair of sets A and B.

4. Vanishing of the Poisson brackets

The theorem is obvious for the case $q = r$. In this section we assume $q \neq r$. We have the following lemmas from I, II, III and I ′, II ′, III ′ of the previous section.

Lemma 1. *There exist* $0 < \gamma$ *arcs* $[A_{j_l}, B_{k_l}] l = 1, 2, \ldots, \gamma$ *with no point of* $a \cup A \cup b \cup B$ *in the arc* $[A_{j_l}, B_{k_l}]$ *other than* A_{j_l}, B_{k_l}.

Proof. We can assume $q < r$ without loss of generality. We see that there is an arc generated by the above $2q+1$ points $a_1, a_2, \ldots, a_{2q+1}$, with at least two points of the above $2r+1$ points $b_1, b_2, \ldots, b_{2r+1}$. Consider the case in which the arc $[a_i, a_{i+1}]$ has two points of b, as $a_j \prec b_k \prec b_{k+1} \prec a_{j+1}$. Then for example if $a_j \prec b_k \prec A_j \prec B_k \prec b_{k+1} \prec a_{j+1}$, the arc $\{A_j, B_k\}$ has no point of $a \cup A \cup b \cup B$ in the arc $[A_j, B_k]$ other than A_j, B_k. For example if $a_j \prec A_j \prec b_k \prec B_k \prec b_{k+1} \prec a_{j+1}$, the arc $[B_{k+1+r}, A_{j+1+q}]$ has no point of $a \cup A \cup b \cup B$ other than B_{k+1+r}, A_{j+1+q}.

We can apply the above argument to the other possible cases. □

Lemma 2. *Each of the points* A_{j_l}, B_{k_l} *belongs to one and only one arc,* $[A_{j_l}, B_{k_l}]$*, for* $l = 1, 2, \ldots, \gamma$.

Lemma 3. *For each* $l = 1, 2, \ldots, \gamma$,

$$\sum_{k=1}^{2r+1} I_{A_{j_l},B_k} + \sum_{k=1}^{2r+1} I_{A'_{j_l},B'_k} = 0, \tag{16}$$

$$\sum_{j=1}^{2q+1} I_{A_j,B_{k_l}} + \sum_{j=1}^{2q+1} I_{A'_j,B'_{k_l}} = 0. \tag{17}$$

Proof. For each $l = 1, 2, \ldots, \gamma$,

$$I_{A_{j_l},B_{k_l}} + I_{A_{j'_l},B_{k'_l}} = 0. \tag{18}$$

We have also

$$\sum_{k \neq k_l} I_{A_{j_l},B_k} = \sum_{k \neq k_l} I_{A'_{j_l},B'_k} = 0, \tag{19}$$

$$\sum_{j \neq j_l} I_{A_j,B_{k_l}} = \sum_{j \neq j_l} I_{A'_j,B'_{k_l}} = 0, \tag{20}$$

which give equations (16) and (17). □

Lemma 4.

(1) The number of points $A_j \in A$, $j \neq j_l$ for $l = 1, 2, \ldots, \gamma$, in the arc $[B_k, b_{k+1}]$, is equal to the number of points of $A_j \in A$, $j \neq j_l$ for $l = 1, 2, \ldots, \gamma$, in the arc $[b_{k+r+1}, B_{k+r+1}]$ for $k = 1, 2, \ldots, 2r + 1$.

(2) The number of points of $B_k \in B$, $k \neq k_l$ for $l = 1, 2, \ldots, \gamma$, in the arc $[A_j, a_{j+1}]$ is equal to the number of points of $B_k \in B$, $k \neq k_l$ for $l = 1, 2, \ldots, \gamma$, in the arc $[a_{j+q+1}, A_{j+q+1}]$ for $j = 1, 2, \ldots, 2q + 1$.

Proof. By considering lemma 2, the above (1) is easily shown for the following two cases:

(1) $B_{k_l} \prec A_{j_l} \prec a_{j_l+1} \prec A_{j_l+1} \prec a_{j_l+2} \cdots \prec a_{j_l+\nu} \prec b_{k_l+1}$;

(2) $B_{k_l} \prec A_{j_l} \prec a_{j_l+1} \prec A_{j_l+1} \prec a_{j_l+2} \cdots \prec a_{j_l+\nu} \prec A_{j_l+\nu} \prec b_{k_l+1}$.

The other cases are also shown in the same way. □

Lemma 5. *For $j = 1, 2, \ldots, 2q + 1$,*

$$\sum_{k \neq k_l \text{ for } l=1,2,\ldots,\gamma} I_{A_j,B_k} = \sum_{k \neq k_l \text{ for } l=1,2,\ldots,\gamma} I_{A'_j,B'_k} = 0. \tag{21}$$

For $k = 1, 2, \ldots, 2r + 1$,

$$\sum_{j \neq j_l \text{ for } l=1,2,\ldots,\gamma} I_{A_j,B_k} = \sum_{j \neq j_l \text{ for } l=1,2,\ldots,\gamma} I_{A'_j,B'_k} = 0. \tag{22}$$

Proof. We see from lemma 4. □

Lemma 6. *For $j = 1, 2, \ldots, 2q + 1$,*

$$\sum_{k=1}^{2r+1} (I_{A_j,B_k} + I_{A'_j,B'_k}) = 0. \tag{23}$$

For $k = 1, 2, \ldots, 2r + 1$,

$$\sum_{j=1}^{2q+1} (I_{A_j,B_k} + I_{A'_j,B'_k}) = 0. \tag{24}$$

Proof. We see from lemmas 3 and 5. □

Lemma 7.

$$\sum_{j=1}^{2q+1} \sum_{k=1}^{2r+1} (I_{A_j,B_k} + I_{A'_j,B'_k}) = 0. \tag{25}$$

Proof. We see directly from lemma 6. □

Let us determine t's by $A_j = W_{t(A_j)}$ and $A'_j = W_{t(A'_j)}$, for $j = 1, 2, \ldots, 2q + 1$, and $B_k = W_{t(B_k)}$ and $B'_k = W_{t(B'_k)}$ for $k = 1, 2, \ldots, 2r + 1$.

Lemma 8. *Let each of the $2s + 1$ points W_i have variable $v_i(t)$ for $i = 1, 2, \ldots, 2s + 1$, which evolves by equation (10) with $m = 2s + 1$. Consider the above pair of sets A and B and its* *conjugate pair of sets A' and B'. Consider the original name of the points in the sets A, B, A' and B'. Then for the Poisson brackets,*

$$\left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)}\right]\right\} = \sum_{l,m=1}^{2s+1} \lambda_{lm} \frac{\partial \exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right]}{\partial v_l} \frac{\partial \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)}\right]}{\partial v_m}, \tag{26}$$

$$\left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)}\right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B'_k)}\right]\right\} = \sum_{l,m=1}^{2s+1} \lambda_{lm} \frac{\partial \exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)}\right]}{\partial v_l} \frac{\partial \exp\left[\sum_{k=1}^{2r+1} v_{t(B'_k)}\right]}{\partial v_m}, \tag{27}$$

we have

$$\left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)}\right]\right\} + \left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)}\right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B'_k)}\right]\right\} = 0 \tag{28}$$

Proof. We see $I_{A_i,A_j} = \lambda_{ij}$ for the i, j, $1, 2, \ldots, 2s+1$. Since

$$\exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)} + \sum_{k=1}^{2r+1} v_{t(B_k)}\right] = \exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)} + \sum_{k=1}^{2r+1} v_{t(B'_k)}\right], \tag{29}$$

by using lemma 7 we have

$$\left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right], \exp\left[\sum_{j=1}^{2r+1} v_{t(B_k)}\right]\right\} + \left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)}\right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B'_k)}\right]\right\}$$

$$= \sum_{l,m=1}^{2s+1} \lambda_{lm} \left(\frac{\partial \exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right]}{\partial v_l} \frac{\partial \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)}\right]}{\partial v_m} \right. \tag{30}$$

$$\left. + \frac{\partial \exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)}\right]}{\partial v_l} \frac{\partial \exp\left[\sum_{k=1}^{2r+1} v_{t(B'_k)}\right]}{\partial v_m} \right) \tag{31}$$

$$= \exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)} + \sum_{k=1}^{2r+1} v_{t(B_k)}\right] \sum_{j=1}^{2q+1} \sum_{k=1}^{2r+1} (I_{A_j,B_k} + I_{A'_j,B'_k}) = 0. \tag{32}$$

□

Proof of theorem. Let T_α be the set of all possible tournaments generated by the $2\alpha+1$ points of the set W for $\alpha = 1, 2, \ldots, s$. Then

$$\{F_q, F_r\} = \left\{\sum_{A\subset T_q} \exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right], \sum_{B\subset T_r} \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)}\right]\right\}$$

$$= \sum_{A\subset T_q} \sum_{B\subset T_r} \left\{\exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)}\right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)}\right]\right\} = 0. \tag{33}$$

Since the Poisson bracket for each pair of the sets $A \in T_q$ and $B \in T_r$, $\left\{ \exp\left[\sum_{j=1}^{2q+1} v_{t(A_j)} \right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B_k)} \right] \right\}$ is canceled by the Poisson bracket $\left\{ \exp\left[\sum_{j=1}^{2q+1} v_{t(A'_j)} \right], \exp\left[\sum_{k=1}^{2r+1} v_{t(B'_k)} \right] \right\}$ for the conjugate pair of sets $A' \in T_q$ and $B' \in T_r$ which is uniquely determined by the pair A and B. □

5. Motivation and discussion

We can generalize the combinatorial method to get the conserved quantities [30] for the general periodic case by Bogoyavlensky [7]. We briefly review our previous study [30] to suggest our next problem. We also show that the form of conserved quantities is convenient to study a stochastic model of interacting particles which is a simplified model for population genetics, ecology and other problems [23, 24, 29, 30]. The form of conserved quantities, originally obtained to calculate the asymptotic probability of coexistence of species, works to study the deterministic approximation of the stochastic model as a nonlinear integrable system, as we could see in the previous sections.

Consider an interacting particle system of n particles for competition among m species, $1, 2, \ldots, m$, whose abundances of particles at time t are $n_1(t), n_2(t), \ldots, n_m(t)$, respectively. For each pair of species a dominance relation is defined. Namely, by an interaction of a particle of species i and a particle of species j, the interacting two particles become two particles of species i with probability $1/2 + a_{ij}$ and become two particles of species j with probability $1/2 + a_{ji}$ where a_{ij} are defined by

$$2\sum_{j=1}^{m} a_{ij} P_j \equiv \sum_{k=0}^{s} P_{i-k} - \sum_{k=0}^{s} P_{i+k}. \tag{34}$$

From the skew symmetry $a_{ij} + a_{ji} = 0$, we see that the total number of particles does not change by interactions, that is to say, $n_1(t) + n_2(t) + \cdots + n_m(t) = n$. A random interaction takes place, where each interacting pair of particles is equally likely to be chosen, in a time interval $\triangle t$.

Note that the case $s = 0$ of our stochastic model gives the Moran version [31] of the Fisher–Wright model for random sampling effect in population genetics.

For infinite n, the relative abundance P_i of species $1, 2, \ldots, m$ is given by

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = P_i \left(\sum_{k=0}^{s} P_{i-k} - \sum_{k=0}^{s} P_{i+k} \right). \tag{35}$$

We introduce our conserved quantities. We say the species i dominates the species j $(j \prec i)$, if $2a_{ij} = 1$. If $2a_{ij} = -1$, we say the species i is dominated by the species j $(i \prec j)$. If $a_{ij} = 0$, we say the species i is neutral to the species j $(i \sim j, j \sim i)$. Hence the relation $i \prec j \prec k$ does not imply $i \prec k$. Consider l species out of the m species. If each of the l species dominates the q species out of the other $l-1$ species, it is dominated by the q species out of the remaining $l-q-1$ species, and is neutral to the other remaining $l-2q-1$ species, then we say that the l species are in a (l, q) regular tournament.

For example, consider the case of seven species with the relations, $i - 3 \sim i, i - 2 \prec i, i - 1 \prec i$, for each species $i = 1, 2, \ldots, 7$. The seven species are in a $(7, 2)$ regular tournament.

Take l particles at random from the system. Let $G_{l,q}$ be the probability that the corresponding species of the l particles are in a (l, q) regular tournament. Then the probability $G_{l,q}$ is the conserved quantity of equation (35),

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{l,q} = 0. \tag{36}$$

For example, when $m = 7$ and $s = 2$, the conserved quantities are $G_{1,0} = P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7$, $G_{2,0} = P_1P_4 + P_2P_5 + P_3P_6 + P_4P_7 + P_5P_1 + P_6P_2 + P_7P_3$, $G_{4,1} = P_1P_2P_4P_6 + P_2P_3P_5P_7 + P_3P_4P_6P_1 + P_4P_5P_7P_2 + P_5P_6P_1P_3 + P_6P_7P_2P_4 + P_7P_1P_3P_5$, $G_{7,2} = P_1P_2P_3P_4P_5P_6P_7$.

When $m = 2s + 1$, $G_{2r+1,r}$, which is equal to G_{2r+1} in section 2, is conserved quantities for $r = 0, 1, \ldots, s$.

To extend our combinatorial method to show the vanishing of the Poisson bracket for general m and s is our next problem.

We briefly explain the stochastic model [23, 24]. Consider the case $m = 3$, $s = 1$ of finite n. Let the abundances of the three types be (n_1, n_2, n_3) at time t and consider the product of the abundances of three species. The abundances at $t + \triangle t$ are

$$\begin{array}{lll}
(n_1 - 1, n_2 + 1, n_3) & \text{with probability} & \dfrac{2n_1n_2}{n(n-1)} \\
(n_1, n_2 - 1, n_3 + 1) & \text{with probability} & \dfrac{2n_2n_3}{n(n-1)} \\
(n_1 + 1, n_2, n_3 - 1) & \text{with probability} & \dfrac{2n_3n_1}{n(n-1)} \\
(n_1, n_2, n_3) & \text{with probability} & \dfrac{n_1(n_1-1) + n_2(n_2-1) + n_3(n_3-1)}{n(n-1)}.
\end{array}$$

So the expected product at $t + \triangle t$ is equal to $\left(1 - 2\frac{\binom{3}{2}}{n(n-1)}\right)n_1n_2n_3$. Consider the probability $G_1(t)$ at time t. For the stochastic process $G_1(t)$, we have the expectation of $G_1(t + \triangle t)$ conditioning on the value $G_1(t)$ as

$$E(G_1(t + \triangle t)|G_1(t)) = \left(1 - 2\frac{\binom{3}{2}}{n(n-1)}\right)G_1(t). \tag{37}$$

For the general case $m = 2s + 1$ we have

$$E(G_r(t + \triangle t)|G_r(t)) = \left(1 - 2\frac{\binom{2r+1}{2}}{n(n-1)}\right)G_r(t) \tag{38}$$

for $r = 0, 1, 2, \ldots, s$. We put $\frac{2}{n-1} = \triangle t$. We then obtain

$$E(G_r(t + \triangle t)|G_r(t)) = \left(1 - \binom{2r+1}{2}\frac{\triangle t}{n}\right)G_r(t) \tag{39}$$

for $r = 0, 1, 2, \ldots, s$. Applying this formula and making use of the argument by Kimura [32] for the Fisher–Wright model, we can show that the asymptotic probability of coexistence of the species, which make regular tournament of order $2r + 1$, is proportional to the expected value of $G_r(t)$ starting from $G_r(0)$ by calculating the second largest eigen value and its eigen vector of the corresponding Markov chain [24].

For the general case of m and s, we have the expected value of $G_{l,q}$ at time $t+u$ conditioning on the value of $G_{l,q}$ at time t approximating $\left(1 - \binom{l}{2}\frac{\triangle t}{n}\right)^{\frac{u}{\triangle t}}$ by $\exp\left(-\binom{l}{2}\frac{u}{n}\right)$ as

$$E(G_{l,q}(t + u)|G_{l,q}(t)) = \exp\left(-\binom{l}{2}\frac{u}{n}\right)G_{l,q}(t), \tag{40}$$

which means the expected $G_{l,q}$ is time invariant for $n = \infty$. From the above we see that our conserved quantities were obtained naturally to study the asymptotic behavior of the stochastic model. We get a system of stochastic differential equations as a diffusion approximation for the change of relative abundances of our stochastic model and obtain an analogous result to equation (40) for $G_{l,q}$ applying the Ito formula [29, 30].

Another possible application of our method will be to the Toda lattice. Henon's method [2] to get conserved quantities is combinatorial. To extend our combinatorial method, we need to define the conjugate pair of sets for the conserved quantities and to show the vanishing of the Poisson bracket for the Toda lattice. This is another next problem for our study.

Acknowledgments

This work was done while the author was at Department of Mathematics, Queen's University, Canada. The author is grateful to Oleg Bogoyavlensky for his hospitality, helpful discussion and comments. The author is also grateful to Rei Inoue for answering the author's questions. Thanks are also due to the referees for suggestions.

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